

# STABLE SCHOTTKY-JACOBI FORMS

JAE-HYUN YANG

**ABSTRACT.** In this article, we prove that there do not exist stable Schottky-Jacobi forms for the universal Jacobian locus and also prove that there exist non-trivial stable Schottky-Jacobi forms for the universal hyperelliptic locus.

## 1. Introduction

For a positive integer  $g$ , we let

$$\mathbb{H}_g = \left\{ \tau \in \mathbb{C}^{(g,g)} \mid \tau = {}^t\tau, \operatorname{Im} \tau > 0 \right\}$$

be the Siegel upper half plane of degree  $g$  and let

$$Sp(2g, \mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^t M J_g M = J_g \}$$

be the symplectic group of degree  $g$ , where  $F^{(k,l)}$  denotes the set of all  $k \times l$  matrices with entries in a commutative ring  $F$  for two positive integers  $k$  and  $l$ ,  ${}^t M$  denotes the transposed matrix of a matrix  $M$  and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then  $Sp(2g, \mathbb{R})$  acts on  $\mathbb{H}_g$  transitively by

$$M \cdot \tau = (A\tau + B)(C\tau + D)^{-1},$$

where  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$  and  $\Omega \in \mathbb{H}_n$ . Let

$$\Gamma_g = Sp(2g, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}) \mid A, B, C, D \text{ integral} \right\}$$

be the Siegel modular group of degree  $g$ . This group acts on  $\mathbb{H}_g$  properly discontinuously.

Let  $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$  be the Siegel modular variety of degree  $g$ , that is, the moduli space of  $g$ -dimensional principally polarized abelian varieties, and let  $\mathcal{M}_g$  be the the moduli space of projective curves of genus  $g$ . Then according to Torelli's theorem, the Jacobi mapping

$$T_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

defined by

$$C \longmapsto J(C) := \text{the Jacobian of } C$$

is injective. The Jacobian locus  $J_g := T_g(\mathcal{M}_g)$  is a  $(3g - 3)$ -dimensional subvariety of  $\mathcal{A}_g$

---

Subject Classification: Primary 14H40, 14H42, 14K25, 32G20

Keywords and phrases: stable Jacobi forms, the Schottky problem, the universal Jacobian locus, stable Schottky-Siegel forms, the universal hyperelliptic locus, stable Schottky-Jacobi forms.

The Schottky problem is to characterize the Jacobian locus or its closure  $\bar{J}_g$  in  $\mathcal{A}_g$ . At first this problem had been investigated from the analytical point of view : to find explicit equations of  $J_g$  (or  $\bar{J}_g$ ) in  $\mathcal{A}_g$  defined by Siegel modular forms on  $\mathbb{H}_g$ , for example, polynomials in the theta constant  $\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}(\tau, 0)$  and their derivatives. The first result in this direction was due to Friedrich Schottky [22] who gave the simple and beautiful equation satisfied by the theta constants of Jacobians of dimension 4. Much later the fact that this equation characterizes the Jacobian locus  $J_4$  was proved by J. Igusa [14] (see also [9], [11] and [13]). Past decades there has been some progress on the characterization of Jacobians by some mathematicians.

For two positive integers  $g$  and  $h$ , we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu^t \lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda')$$

with  $(\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)}$ . We define the *Jacobi group*  $G^J$  of degree  $g$  and index  $h$  that is the semidirect product of  $Sp(2g, \mathbb{R})$  and  $H_{\mathbb{R}}^{(g,h)}$

$$G^J = Sp(2g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}^t \mu' - \tilde{\mu}^t \lambda'))$$

with  $M, M' \in Sp(2g, \mathbb{R})$ ,  $(\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)}$  and  $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$ . Then  $G^J$  acts on  $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$  transitively by

$$(M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = \left( M \cdot \Omega, (Z + \lambda \Omega + \mu)(C\Omega + D)^{-1} \right),$$

where  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$  and  $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$ . We note that the Jacobi group  $G^J$  is *not* a reductive Lie group and the homogeneous space  $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$  is not a symmetric space. From now on, for brevity we write  $\mathbb{H}_{g,h} = \mathbb{H}_g \times \mathbb{C}^{(h,g)}$ . The homogeneous space  $\mathbb{H}_{g,h}$  is called the *Siegel-Jacobi space* of degree  $g$  and index  $h$ .

Let  $\Gamma_g^J := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$  be the Jacobi modular group. Let

$$\mathcal{A}_{g,h} := \Gamma_g^J \backslash \mathbb{H}_{g,h}$$

be the universal abelian variety. Consider the natural projection map

$$\pi_{g,h} : \mathcal{A}_{g,h} \longrightarrow \mathcal{A}_g.$$

Let

$$J_{g,h} := \pi_{g,h}^{-1}(J_g)$$

be the universal Jacobian locus and let

$$Hyp_{g,h} := \pi_{g,h}^{-1}(Hyp_g)$$

be the universal hyperelliptic locus, where  $Hyp_g$  is the hyperelliptic locus in  $\mathcal{A}_g$ .

Let  $2\mathcal{M}$  be a positive definite, even unimodular integral symmetric matrix of degree  $h$ . According to Theorem 3.6 in [27], if  $g + h > 2k + 1$  with a nonnegative integer  $k$ , the Siegel-Jacobi operator

$$\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

is an isomorphism (see also Theorem 2.2). Using this fact, we define the notion of stable Jacobi forms of weight  $k$  and index  $\mathcal{M}$ . A Jacobi form  $F \in J_{k,\mathcal{M}}(\Gamma_g)$  is said to be a *Schottky-Jacobi form* for  $J_{g,h}$  (resp.  $Hyp_{g,h}$ ) if it vanishes along  $J_{g,h}$  (resp.  $Hyp_{g,h}$ ). In a natural way, we can define the notion of *stable Schottky-Jacobi forms* of index  $\mathcal{M}$ . For precise definitions, we refer to Definition 2.3 and Definition 4.2.

The aim of this paper is to prove the non-existence of stable Schottky-Jacobi forms for the universal Jacobian locus and also to prove that there exist non-trivial stable Schottky-Jacobi forms for the universal hyperelliptic locus.

This article is organized as follows. In Section 2, we review some properties of the Siegel-Jacobi operator and the notion of stable Jacobi forms introduced by J.-H. Yang [30]. In Section 3, we review the notion of stable Schottky-Siegel forms and the works that were done recently by G. Codogni and N. I. Shepherd-Barron [3, 4]. In Section 4, we introduce the notion of stable Schottky-Jacobi forms and prove the following two theorems.

**Theorem 1.1.** *Let  $2\mathcal{M}$  be a positive definite, even unimodular integral symmetric matrix of degree  $h$ . Then there do not exist stable Schottky-Jacobi forms of index  $\mathcal{M}$  for the universal Jacobian locus.*

**Theorem 1.2.** *Let  $2\mathcal{M}$  be a positive definite, even unimodular integral symmetric matrix of degree  $h$ . Then there exist non-trivial stable Schottky-Jacobi forms of index  $\mathcal{M}$  for the universal hyperelliptic locus.*

In the final section, we make some comments and present several questions.

**Notations:** We denote by  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by  $\mathbb{Z}$  and  $\mathbb{Z}^+$  the ring of integers and the set of all positive integers respectively.  $\mathbb{R}^+$  denotes the set of all positive real numbers.  $\mathbb{Z}_+$  and  $\mathbb{R}_+$  denote the set of all nonnegative integers and the set of all nonnegative real numbers respectively. The symbol “:=” means that the expression on the right is the definition of that on the left. For two positive integers  $k$  and  $l$ ,  $F^{(k,l)}$  denotes the set of all  $k \times l$  matrices with entries in a commutative ring  $F$ . For a square matrix  $A \in F^{(k,k)}$  of degree  $k$ ,  $\sigma(A)$  denotes the trace of  $A$ . For any  $M \in F^{(k,l)}$ ,  ${}^tM$  denotes the transpose of a matrix  $M$ .  $I_n$  denotes the identity matrix of degree  $n$ . We put  $i = \sqrt{-1}$ .

## 2. Stable Jacobi Forms

For a non-negative integer  $k$ , we denote by  $[\Gamma_g, k]$  the vector space of all Siegel modular forms of weight  $k$ . The Siegel  $\Phi$ -operator

$$\Phi_g : [\Gamma_g, k] \longrightarrow [\Gamma_{g-1}, k]$$

is an important linear map defined by

$$(2.1) \quad (\Phi_g f)(\tau) := \lim_{t \rightarrow \infty} f \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, \quad f \in [\Gamma_g, k], \tau \in \mathbb{H}_{g-1}.$$

H. Maass [18] proved that if  $k$  is even and  $k > 2g$ , then  $\Phi_g$  is surjective. E. Freitag [7] proved that if  $g > 2k$ , then  $\Phi_g$  is injective. Using the theory of singular modular forms developed by Freitag [8, 10], he showed the following:

$$(SO1) \quad [\Gamma_g, k] = 0 \quad \text{for } g > 2k, \ k \not\equiv 0 \pmod{4}.$$

$$(SO2) \quad \Phi_g \text{ is an isomorphism if } g > 2k + 1.$$

**Definition 2.1.** A collection  $(f_g)_{g \geq 0}$  is called a *stable modular form of weight  $k$*  if it satisfies the following conditions (SM1) and (SM2):

$$(SM1) \quad f_g \in [\Gamma_g, k] \text{ for all } g \geq 0.$$

$$(SM2) \quad \Phi_g f_g = f_{g-1} \text{ for all } g > 0.$$

Let  $\rho$  be a rational representation of  $GL(g, \mathbb{C})$  on a finite dimensional complex vector space  $V_\rho$ . Let  $\mathcal{M} \in \mathbb{R}^{(h, h)}$  be a symmetric half-integral semi-positive definite matrix of degree  $h$ . The canonical automorphic factor

$$J_{\rho, \mathcal{M}} : G^J \times \mathbb{H}_{g, h} \longrightarrow GL(V_\rho)$$

for  $G^J$  on  $\mathbb{H}_{g, h}$  is given as follows :

$$\begin{aligned} J_{\rho, \mathcal{M}}((M, (\lambda, \mu; \kappa)), (\tau, z)) &= e^{2\pi i \sigma(\mathcal{M}(z + \lambda\tau + \mu)(C\tau + D)^{-1}C^t(z + \lambda\tau + \mu))} \\ &\quad \times e^{-2\pi i \sigma(\mathcal{M}(\lambda\tau^t\lambda + 2\lambda^t z + \kappa + \mu^t\lambda))} \rho(C\tau + D), \end{aligned}$$

where  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g, h)}$  and  $(\tau, z) \in \mathbb{H}_{g, h}$ . We refer to [29] for a geometrical construction of  $J_{\rho, \mathcal{M}}$ .

Let  $C^\infty(\mathbb{H}_{g, h}, V_\rho)$  be the algebra of all  $C^\infty$  functions on  $\mathbb{H}_{g, h}$  with values in  $V_\rho$ . For  $f \in C^\infty(\mathbb{H}_{g, h}, V_\rho)$ , we define

$$\begin{aligned} (f|_{\rho, \mathcal{M}}[(M, (\lambda, \mu; \kappa))])(\tau, z) &= J_{\rho, \mathcal{M}}((M, (\lambda, \mu; \kappa)), (\tau, z))^{-1} \\ &\quad f((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1}), \end{aligned}$$

where  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R})$ ,  $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g, h)}$  and  $(\tau, z) \in \mathbb{H}_{g, h}$ .

**Definition 2.2.** Let  $\rho$  and  $\mathcal{M}$  be as above. Let

$$H_{\mathbb{Z}}^{(g, h)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g, h)} \mid \lambda, \mu, \kappa \text{ integral} \right\}$$

be the discrete subgroup of  $H_{\mathbb{R}}^{(g, h)}$ . A *Jacobi form of index  $\mathcal{M}$*  with respect to  $\rho$  on a subgroup  $\Gamma$  of  $\Gamma_g$  of finite index is a holomorphic function  $f \in C^\infty(\mathbb{H}_{g, h}, V_\rho)$  satisfying the following conditions (A) and (B):

$$(A) \quad f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f \text{ for all } \tilde{\gamma} \in \tilde{\Gamma} := \Gamma \ltimes H_{\mathbb{Z}}^{(g, h)}.$$

(B) For each  $M \in \Gamma_g$ ,  $f|_{\rho, \mathcal{M}}[M]$  has a Fourier expansion of the following form :

$$(f|_{\rho, \mathcal{M}}[M])(\tau, z) = \sum_{\substack{T=tT \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}(g, h)} c(T, R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(T\tau)} \cdot e^{2\pi i \sigma(Rz)}$$

with  $\lambda_\Gamma (\neq 0) \in \mathbb{Z}$  and  $c(T, R) \neq 0$  only if  $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^t R & \mathcal{M} \end{pmatrix} \geq 0$ .

If  $g \geq 2$ , the condition (B) is superfluous by Köcher principle (cf. [33] Lemma 1.6). We denote by  $J_{\rho, \mathcal{M}}(\Gamma)$  the vector space of all Jacobi forms of index  $\mathcal{M}$  with respect to  $\rho$  on  $\Gamma$ . Ziegler (cf. [33] Theorem 1.8 or [6] Theorem 1.1) proved that the vector space  $J_{\rho, \mathcal{M}}(\Gamma)$  is finite dimensional. In the special case  $\rho(A) = (\det(A))^k$  with  $A \in GL(g, \mathbb{C})$  and a fixed  $k \in \mathbb{Z}$ , we write  $J_{k, \mathcal{M}}(\Gamma)$  instead of  $J_{\rho, \mathcal{M}}(\Gamma)$  and call  $k$  the *weight* of the corresponding Jacobi forms. For more results about Jacobi forms with  $g > 1$  and  $h > 1$ , we refer to [26, 27, 28, 29, 30, 31, 32] and [33]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree  $2g$  (cf. [16, 17]).

Now we consider the special case  $\rho = \det^k$  with  $k \in \mathbb{Z}_+$ . We define the Siegel-Jacobi operator

$$\Psi_{g, \mathcal{M}} : J_{k, \mathcal{M}}(\Gamma_g) \longrightarrow J_{k, \mathcal{M}}(\Gamma_{g-1})$$

by

$$(2.2) \quad (\Psi_{g, \mathcal{M}} F)(\tau, z) := \lim_{t \rightarrow \infty} F \left( \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0) \right),$$

where  $F \in J_{k, \mathcal{M}}(\Gamma_g)$ ,  $\tau \in \mathbb{H}_{g-1}$  and  $z \in \mathbb{C}^{(h, g-1)}$ . We observe that the above limit exists and  $\Psi_{g, \mathcal{M}}$  is a well-defined linear map (cf. [33]).

J.-H. Yang [27] proved the following theorems.

**Theorem 2.1.** *Let  $2\mathcal{M}$  be a positive even unimodular symmetric integral matrix of degree  $h$  and let  $k$  be an even nonnegative integer. If  $g + h > 2k$ , then the Siegel-Jacobi operator  $\Psi_{g, \mathcal{M}}$  is injective.*

*Proof.* See Theorem 3.5 in [27]. □

**Theorem 2.2.** *Let  $2\mathcal{M}$  be as above in Theorem 2.1 and let  $k$  be an even nonnegative integer. If  $g + h > 2k + 1$ , then the Siegel-Jacobi operator  $\Psi_{g, \mathcal{M}}$  is an isomorphism.*

*Proof.* See Theorem 3.6 in [27]. □

**Remark 2.1.** A Jacobi form in  $J_{k, \mathcal{M}}(\Gamma_g)$  is said to be **singular** if it admits a Fourier expansion such that a Fourier coefficient  $c(T, R)$  vanishes unless

$$\det \begin{pmatrix} T & \frac{1}{2} R \\ \frac{1}{2} {}^t R & \mathcal{M} \end{pmatrix} = 0.$$

Let  $2\mathcal{M}$  be as above in Theorem 2.1. Yang proved that if  $k$  is an even nonnegative integer and  $g + \text{rank}(\mathcal{M}) > 2k$ , then any non-zero Jacobi form in  $J_{k, \mathcal{M}}(\Gamma_g)$  is singular (cf. [28, Theorem 4.5]).

**Theorem 2.3.** *Let  $2\mathcal{M}$  be as above in Theorem 2.1 and let  $k$  be an even nonnegative integer. If  $2k > 4g + h$ , then the Siegel-Jacobi operator  $\Psi_{g, \mathcal{M}}$  is surjective.*

*Proof.* See Theorem 3.7 in [27]. □

**Remark 2.2.** Yang [27, Theorem 4.2] proved that the action of the Hecke operators on Jacobi forms is compatible with that of the Siegel-Jacobi operator.

**Definition 2.3.** A collection  $(F_g)_{g \geq 0}$  is called a **stable Jacobi form** of weight  $k$  and index  $\mathcal{M}$  if it satisfies the following conditions (SJ1) and (SJ2):

$$(SJ1) \quad F_g \in J_{k, \mathcal{M}}(\Gamma_g) \quad \text{for all } g \geq 0.$$

$$(SJ2) \quad \Psi_{g, \mathcal{M}} F_g = F_{g-1} \quad \text{for all } g \geq 1.$$

**Remark 2.3.** The concept of a stable Jacobi forms was introduced by Yang [30].

**Example.** Let  $S$  be a positive even unimodular symmetric integral matrix of degree  $2k$  and let  $c \in \mathbb{Z}^{(2k, h)}$  be an integral matrix. We define the theta series  $\vartheta_{S, c}^{(g)}$  by

$$\vartheta_{S, c}^{(g)}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^{(2k, g)}} e^{\pi i \{ \sigma(S \lambda \tau \ ^t \lambda) + 2\sigma(\ ^t c S \lambda \ ^t z) \}}, \quad (\tau, z) \in \mathbb{H}_{g, h}.$$

It is easily seen that  $\vartheta_{S, c}^{(g)} \in J_{k, \mathcal{M}}(\Gamma_g)$  with  $\mathcal{M} := \frac{1}{2} \ ^t c S c$  for all  $g \geq 0$  and  $\Psi_{g, \mathcal{M}} \vartheta_{S, c}^{(g)} = \vartheta_{S, c}^{(g-1)}$  for all  $g \geq 1$ . Thus the collection

$$\Theta_{S, c} := \left( \vartheta_{S, c}^{(g)} \right)_{g \geq 0}$$

is a stable Jacobi form of weight  $k$  and index  $\mathcal{M}$ .

### 3. Stable Schottky-Siegel Forms

Let  $\mathcal{A}_g^{\text{Sat}}$  be the Satake compactification of the Siegel modular variety  $\mathcal{A}_g$  (cf. [21]).

$$\mathcal{A}_g^{\text{Sat}} = \mathcal{A}_g \cup \mathcal{A}_{g-1} \cup \cdots \cup \mathcal{A}_1 \cup \mathcal{A}_0.$$

W. Baily [1] proved that  $\mathcal{A}_g^{\text{Sat}}$  is a normal projective variety in which  $\mathcal{A}_g$  is Zariski open. In particular, we have a closed embedding

$$\iota_g : \mathcal{A}_{g-1}^{\text{Sat}} \hookrightarrow \mathcal{A}_g^{\text{Sat}}.$$

The collection  $(\mathcal{A}_g^{\text{Sat}})_{g \geq 0}$  and the above embeddings  $(\iota_g)_{g \geq 0}$  define the projective limit

$$\mathcal{A}_\infty^{\text{Sat}} := \bigcup_{g \geq 0} \mathcal{A}_g^{\text{Sat}} = \varprojlim_g \mathcal{A}_g^{\text{Sat}}$$

which is called the *stable Satake compactification*. Let  $\mathcal{L}_g$  be the determinant line bundle of the Hodge bundle over  $\mathcal{A}_g$ . Then we have the isomorphism

$$H^0(\mathcal{A}_g, \mathcal{L}_g^{\otimes k}) \cong [\Gamma_g, k].$$

Let  $J_g^{\text{Sat}}$  (resp.  $\text{Hyp}_g^{\text{Sat}}$ ) be the closure of  $J_g$  (resp.  $\text{Hyp}_g$ ) inside  $\mathcal{A}_g^{\text{Sat}}$ . We define

$$J_\infty := \bigcup_{g \geq 0} J_g^{\text{Sat}} \quad \text{and} \quad \text{Hyp}_\infty := \bigcup_{g \geq 0} \text{Hyp}_g^{\text{Sat}}.$$

**Definition 3.1.** A pair  $(\Lambda, Q)$  is called a quadratic form if  $\Lambda$  is a lattice and  $Q$  is an integer-valued bilinear symmetric form on  $\Lambda$ . The rank of  $(\Lambda, Q)$  is defined to be the rank of  $\Lambda$ . For  $v \in \Lambda$ , the integer  $Q(v, v)$  is called the norm of  $v$ . A quadratic form  $(\Lambda, Q)$  is said to be even if  $Q(v, v)$  is even for all  $v \in \Lambda$ . A quadratic form  $(\Lambda, Q)$  is said to be unimodular if  $\det(Q) = 1$ .

**Definition 3.2.** Let  $(\Lambda, Q)$  be an even unimodular positive definite quadratic form of rank  $m$ . For a positive integer  $g$ , the theta series  $\theta_{Q,g}$  associated to  $(\Lambda, Q)$  is defined to be

$$\theta_{Q,g}(\tau) := \sum_{x_1, \dots, x_g \in \Lambda} \exp \left( \pi i \sum_{p,q=1}^g Q(x_p, x_q) \tau_{pq} \right), \quad \tau = (\tau_{pq}) \in \mathbb{H}_g.$$

It is well known that  $\theta_{Q,g}(\tau)$  is a Siegel modular form on  $\mathbb{H}_g$  of weight  $\frac{m}{2}$ . We easily see that

$$\Phi_{g+1}(\theta_{Q,g+1}) = \theta_{Q,g} \quad \text{for all } g \geq 0.$$

Therefore the collection of all theta series associated to  $(\Lambda, Q)$

$$(3.1) \quad \Theta_Q := (\theta_{Q,g})_{g \geq 0}$$

is a stable modular form of weight  $\frac{m}{2}$ .

Freitag [8] proved the following theorem.

**Theorem 3.1.** The ring of stable modular forms is a polynomial ring in countably many theta series  $\Theta_Q = (\theta_{Q,g})_{g \geq 0}$  associated to irreducible positive even unimodular quadratic forms.

*Proof.* See Theorem 2.5 in [8]. □

**Definition 3.3.** A modular form  $f \in [\Gamma_g, k]$  is called a Schottky-Siegel form of weight  $k$  for  $J_g$  (resp.  $\text{Hyp}_g$ ) if it vanishes along  $J_g$  (resp.  $\text{Hyp}_g$ ). A collection  $(f_g)_{g \geq 0}$  is called a stable Schottky-Siegel form of weight  $k$  for the Jacobian locus (resp. the hyperelliptic locus) if  $(f_g)_{g \geq 0}$  is a stable modular form of weight  $k$  and  $f_g$  vanishes along  $J_g$  (resp.  $\text{Hyp}_g$ ) for every  $g \geq 0$ .

G. Codogni and N. I. Shepherd-Barron [4] proved the following.

**Theorem 3.2.** There do not exist stable Schottky-Siegel form for the Jacobian locus.

*Proof.* See Theorem 1.3 and Corollary 1.4 in [4]. □

**Remark 3.1.** Let

$$(3.2) \quad \varphi_g(\tau) := \theta_{E_8 \oplus E_8, g}(\tau) - \theta_{D_{16}^+, g}(\tau), \quad \tau \in \mathbb{H}_g$$

be the Igusa modular form, that is, the difference of the theta series in genus  $g$  associated to the two distinct positive even unimodular quadratic forms  $E_8 \oplus E_8$  and  $D_{16}^+$  of rank 16. We see that  $\varphi_g(\tau)$  is a Siegel modular form on  $\mathbb{H}_g$  of weight 8. Since  $\Phi_g \varphi_g = \varphi_{g-1}$  for all  $g \geq 1$ , a collection  $(\varphi_g)_{g \geq 0}$  is a stable modular form of weight 8. Igusa [14, 15] showed that the Schottky-Siegel form discovered by Schottky [22] is an explicit rational multiple of  $\varphi_4$ . In [14], he also showed that the Jacobian locus  $J_4$  is reduced and irreducible, and so cuts out exactly  $J_4$  in  $\mathcal{A}_4$ . Indeed,  $\varphi_4(\tau)$  is a degree 16 polynomial in the Thetanullwerte of genus 4. On the other hand, Grushevsky and Salvati Manni [12] showed that the Igusa modular

form  $\varphi_5$  of genus 5 cuts out exactly the trigonal locus in  $J_5$  and so does not vanish along  $J_5$ . Thus  $(\varphi_g)_{g \geq 0}$  is not a stable Schottky-Siegel form.

G. Codogni [3] proved the following.

**Theorem 3.3.** *There exist non-trivial stable Schottky-Siegel form for the hyperelliptic locus. Precisely the ideal of stable Schottky-Siegel forms for the hyperelliptic locus is generated by differences of theta series*

$$\Theta_P - \Theta_Q,$$

where  $P$  and  $Q$  are positive definite even unimodular quadratic forms of the same rank.

*Proof.* See Theorem 1.2 in [3]. □

**Remark 3.2.** *Let  $P$  and  $Q$  be two positive even unimodular quadratic forms of the same rank. We let*

$$\Theta_P := (\theta_{P,g})_{g \geq 0} \quad \text{and} \quad \Theta_Q := (\theta_{Q,g})_{g \geq 0}$$

*be two stable modular forms. Codogni [3, Theorem 1.4] showed that the difference of theta series*

$$\Theta_P - \Theta_Q$$

*is a stable Schottky-Siegel form for the hyperelliptic locus when one of the following conditions (1)–(3):*

- (1)  $\text{rank}(P) = \text{rank}(Q) = 24$  and the two quadratic forms have the same number of vectors of norm 2;
- (2)  $\text{rank}(P) = \text{rank}(Q) = 32$  and the two quadratic forms do not have any vectors of norm 2;
- (3)  $\text{rank}(P) = \text{rank}(Q) = 48$  and the two quadratic forms do not have any vectors of norm 2 and 4.

#### 4. Stable Schottky-Jacobi Forms and Proofs of Main Theorems

In this section, we introduce the notion of stable Schottky-Jacobi forms and prove the main theorems.

We let

$$\mathcal{A}_{g,h} := \Gamma_{g,h}^J \backslash \mathbb{H}_{g,h}$$

be the universal abelian variety and let

$$\mathcal{A}_{g,h}^{\text{Sat}} := \mathcal{A}_{g,h} \cup \mathcal{A}_{g-1,h} \cup \cdots \cup \mathcal{A}_{1,h} \cup \mathcal{A}_{0,h}$$

be the Satake compactification of  $\mathcal{A}_{g,h}$ . We consider the natural projection map

$$\pi_{g,h} : \mathcal{A}_{g,h} \longrightarrow \mathcal{A}_g$$

of  $\mathcal{A}_{g,h}$  onto  $\mathcal{A}_g$ . Let

$$J_{g,h} := \pi_{g,h}^{-1}(J_g)$$

be the universal Jacobian locus and let

$$\text{Hyp}_{g,h} := \pi_{g,h}^{-1}(\text{Hyp}_g)$$



be the universal hyperelliptic locus. Let  $J_{g,h}^S$  (resp.  $\text{Hyp}_{g,h}^S$ ) be the closure of  $J_{g,h}$  (resp.  $\text{Hyp}_{g,h}$ ) in  $\mathcal{A}_{g,h}^{\text{Sat}}$ . We put

$$\mathcal{A}_{\infty,h} := \bigcup_{g \geq 0} \mathcal{A}_{g,h}^{\text{Sat}},$$

$$J_{\infty,h} := \bigcup_{g \geq 0} J_{g,h}^S$$

and

$$\text{Hyp}_{\infty,h} := \bigcup_{g \geq 0} \text{Hyp}_{g,h}^S.$$

**Definition 4.1.** Let  $\mathcal{M}$  be a half-integral semi-positive symmetric matrix of degree  $h$  and  $k \in \mathbb{Z}_+$ . A Jacobi form  $F \in J_{k,\mathcal{M}}(\Gamma_g)$  is called a **Schottky-Jacobi form** of weight  $k$  and index  $\mathcal{M}$  for the universal Jacobian (resp. hyperelliptic) locus if it vanishes along  $J_{g,h}$  (resp.  $\text{Hyp}_{g,h}$ ).

**Definition 4.2.** Let  $\mathcal{M}$  be a half-integral semi-positive symmetric matrix of degree  $h$  and  $k \in \mathbb{Z}_+$ . A collection  $(F_g)_{g \geq 0}$  is called a **stable Schottky-Jacobi form** of weight  $k$  and index  $\mathcal{M}$  if it satisfies the following conditions (SSJ1) and (SSJ2):

- (SSJ1)  $F_g \in J_{k,\mathcal{M}}(\Gamma_g)$  is a Schottky-Jacobi form of weight  $k$  and index  $\mathcal{M}$  for all  $g \geq 0$ .
- (SSJ2)  $\Psi_{g,\mathcal{M}} F_g = F_{g-1}$  for all  $g \geq 1$ .

**Theorem 4.1.** Let  $2\mathcal{M}$  be a positive even unimodular symmetric integral matrix of degree  $h$ . Then there do not exist stable Schottky-Jacobi forms of index  $\mathcal{M}$  for the universal Jacobian locus.

*Proof.* We first observe that  $h \equiv 0 \pmod{8}$  (cf. [23]). Assume that there exists a non-trivial stable Schottky-Jacobi form  $(F_g)_{g \geq 0}$  of weight  $k$  and index  $\mathcal{M}$  for the universal Jacobian locus.

**Case 1:**  $k$  is even.

Using the Shimura isomorphism (cf. [24, 25]), we obtain the following

$$(4.1) \quad J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k_*] \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z),$$

where  $k_* := k - \frac{h}{2}$  and

$$(4.2) \quad \vartheta_{2\mathcal{M}}^{[g]}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^{(h,g)}} e^{2\pi i \sigma(\mathcal{M}(\lambda \tau^t \lambda + 2\lambda^t z))}.$$

We refer to [33, Theorem 3.3] for the proof of the formula (4.1). We see from (SO1) in Section 2 that  $[\Gamma_g, k_*] = 0$  if  $g + h > 2k$  and  $k \not\equiv 0 \pmod{4}$ . So  $k \equiv 0 \pmod{4}$ . We observe that the Siegel-Jacobi operator  $\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \rightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$  is an isomorphism if  $g + h > 2k + 1$  (see Theorem 2.2 in Section 2). It is easy to see that

$$\Psi_{g,\mathcal{M}} \vartheta_{2\mathcal{M}}^{[g]} = \vartheta_{2\mathcal{M}}^{[g-1]} \quad \text{for all } g \geq 1.$$

According to the formula (4.1), we may write

$$F_g(\tau, z) = f_g(\tau) \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z), \quad f \in [\Gamma, k_*].$$

Now we have, for  $(\tau, z) \in \mathbb{H}_{g-1, h}$ ,

$$\begin{aligned} (\Psi_{g, \mathcal{M}} F_g)(\tau, z) &= \lim_{t \rightarrow \infty} F_g \left( \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0) \right) \\ &= \lim_{t \rightarrow \infty} f_g \left( \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix} \right) \cdot \vartheta_{2\mathcal{M}}^{[g]} \left( \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z, 0) \right) \\ &= (\Phi_g f_g)(\tau) \cdot \vartheta_{2\mathcal{M}}^{[g-1]}(\tau, z). \end{aligned}$$

Here  $\Phi_g$  is the Siegel  $\Phi$ -operator defined by (2.1).

On the other hand, by the assumption that  $(F_g)_{g \geq 0}$  is a stable Schottky-Jacobi form, we have

$$\Psi_{g, \mathcal{M}} F_g = F_{g-1} = f_{g-1} \cdot \vartheta_{2\mathcal{M}}^{[g-1]} \quad \text{for some } f_{g-1} \in [\Gamma_{g-1}, k]$$

for all  $g \geq 1$ . Therefore

$$\Phi_g f_g = f_{g-1} \quad \text{for all } g \geq 1.$$

Obviously  $f_g$  vanishes along  $J_g$  for all  $g \geq 0$ . Thus  $(f_g)_{g \geq 0}$  is a non-trivial stable Schottky-Siegel form of weight  $k_*$ . This contradicts the non-existence of a non-trivial stable Schottky-Siegel form for the Jacobian locus (see Theorem 3.2).

**Case 2:**  $k$  is odd.

Using the Shimura isomorphism, we may write

$$F_g(\tau, z) = \psi_g(\tau) \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z) \quad \text{for all } g \geq 1,$$

where  $\vartheta_{2\mathcal{M}}^{[g]}(\tau, z)$  is the theta series defined by Formula (4.2) and  $f_g(\tau)$  satisfies the following behaviours (4.3) and (4.4):

$$(4.3) \quad \psi_g(\tau + S) = \psi_g(\tau) \quad \text{for all } S = {}^t S \in \mathbb{Z}^{(g, g)};$$

$$(4.4) \quad \psi_g(-\tau^{-1}) = \det(-\tau)^k \det\left(\frac{\tau}{i}\right)^{-\frac{h}{2}} \psi_g(\tau), \quad \tau \in \mathbb{H}_g.$$

We put

$$\xi_g(\tau) := \{\psi_g(\tau)\}^2 \quad \text{for all } g \geq 1.$$

Then we see easily that a collection  $\xi = (\xi_g)_{g \geq 0}$  is a non-trivial stable Schottky-Siegel form of weight  $2k - h$  for the Jacobian locus. This contradicts the non-existence of a non-trivial stable Schottky-Siegel form for the Jacobian locus. Hence we complete the proof of the above theorem(=Theorem 1.1).  $\square$

**Theorem 4.2.** *Let  $2\mathcal{M}$  be a positive even unimodular symmetric integral matrix of degree  $h$ . Then there exist non-trivial stable Schottky-Jacobi forms of  $\mathcal{M}$  for the universal hyperelliptic locus.*

*Proof.* According to Theorem 3.3, there exists a non-trivial stable Schottky-Siegel form  $(f_g)_{g \geq 0}$  of weight  $k$  for the hyperelliptic locus. We see from (SO1) in Section 2 that  $k \equiv 0 \pmod{4}$  and  $k \in \mathbb{Z}^+$ . We put  $\ell := k + \frac{h}{2}$ . Then using the Shimura isomorphism, we have

$$J_{\ell, \mathcal{M}}(\Gamma_g) = [\Gamma_g, k] \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z),$$

where  $\vartheta_{2\mathcal{M}}^{[g]}(\tau, z)$  is the theta series defined by Formula (4.2). We define the Jacobi forms

$$F_g(\tau, z) := f_g(\tau) \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z), \quad g \geq 0.$$

Since  $f_g \in [\Gamma_g, k]$  is a Jacobi form of weight  $k$  and index 0, we get  $F_g \in J_{\ell, \mathcal{M}}(\Gamma_g)$  for all  $g \geq 0$ . For  $[(\tau, z)] \in \text{Hyp}_{g, h}$ ,

$$F_g(\tau, z) = 0, \quad g \geq 0.$$

By a simple calculation, we obtain

$$\Psi_{g, \mathcal{M}} F_g = F_{g-1} \quad \text{for all } g \geq 1.$$

Thus  $(F_g)_{g \geq 0}$  is a non-trivial stable Schottky-Jacobi form of weight  $\ell$  and index  $\mathcal{M}$  for the universal hyperelliptic locus  $\text{Hyp}_{\infty, h}$ . This completes the proof of the above theorem(=Theorem 1.2).  $\square$

We define the invariant  $\mu(Q)$  of a quadratic form  $(Q, \Lambda)$  by

$$\mu(Q) := \min\{Q(v, v) \mid v \in \Lambda, v \neq 0\}.$$

**Theorem 4.3.** *Let  $2\mathcal{M}$  be a positive even unimodular symmetric integral matrix of degree  $h$ . Let  $(Q, \Lambda)$  and  $(P, \Gamma)$  be two positive even unimodular quadratic forms of rank  $m$ . Assume that*

$$\frac{m}{\mu} \leq 8, \quad \text{where } \mu := \min\{\mu(Q), \mu(P)\}.$$

We put

$$F_g(\tau, z) := \{\theta_{Q, g}(\tau) - \theta_{P, g}(\tau)\} \cdot \vartheta_{2\mathcal{M}}^{(g)}(\tau, z), \quad g \geq 0.$$

Then  $(F_g)_{g \geq 0}$  is a stable Schottky-Jacobi form of weight  $\frac{1}{2}(m + h)$  and index  $\mathcal{M}$  for the universal hyperelliptic locus.

*Proof.* It is easily seen that

$$(4.5) \quad \theta_{Q, g}, \theta_{P, g} \in J_{\frac{m}{2}, 0}(\Gamma_g) \quad \text{and} \quad \theta_{Q, g} \cdot \vartheta_{2\mathcal{M}}^{[g]}, \theta_{P, g} \cdot \vartheta_{2\mathcal{M}}^{[g]} \in J_{\frac{1}{2}(m+h), \mathcal{M}}(\Gamma_g)$$

for all  $g \geq 0$ . The proof follows immediately from Theorem 5.5 in [3] and the above facts (4.5).  $\square$

## 5. Final Remarks

In the final section, we make some remarks and present several open questions.

**Remark 5.1.** *Let  $2\mathcal{M}$  be a positive even unimodular symmetric integral matrix of degree  $h$ . Assume that*

$$g + \text{rank}(\mathcal{M}) > 2k + 1 \quad \text{and } k \in \mathbb{Z}_+ \text{ is even.}$$

We denote by  $\mathcal{C}_{k, \mathcal{M}}$  be the vector space of stable Jacobi forms of weight  $k$  and index  $\mathcal{M}$ . According to Theorem 2.2, the Siegel-Jacobi operator  $\Psi_{g, \mathcal{M}} : J_{k, \mathcal{M}}(\Gamma_g) \longrightarrow J_{k, \mathcal{M}}(\Gamma_{g-1})$  is an isomorphism, and hence we obtain

$$\dim \mathcal{C}_{k, \mathcal{M}} = \dim J_{k, \mathcal{M}}(\Gamma_g).$$

From Formula (4.1), we see that

$$J_{k, \mathcal{M}}(\Gamma_g) = [\Gamma_g, k_*] \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z), \quad \text{where } k_* := k - \frac{h}{2}.$$

Therefore from (SO1) in Section 2, we get the vanishing result:

$$J_{k,\mathcal{M}}(\Gamma_g) = 0 \quad \text{if } 2k \not\equiv h \pmod{8}.$$

Thus  $k \equiv 0 \pmod{4}$  if  $J_{k,\mathcal{M}}(\Gamma_g) \neq 0$ . According to Yang [28], any Jacobi form in  $J_{k,\mathcal{M}}(\Gamma_g)$  is singular. We note that any element in  $[\Gamma_g, k - \frac{h}{2}]$  is a singular modular form (see [8, 10]). Hence we conclude that  $\mathcal{C}_{k,\mathcal{M}}$  is spanned by stable Jacobi forms of the form

$$\left( \theta_{P,g}(\tau) \vartheta_{2\mathcal{M}}^{[g]}(\tau, z) \right)_{g \geq 0},$$

where  $P$  runs over the set of positive even unimodular quadratic forms of rank  $2k - h$ .

**Remark 5.2.** Let  $\varphi_g(\tau)$  be the Igusa modular form defined by the formula (3.2). We denote by  $[\Gamma_g, k]_0$  be the space of all Siegel cuspidal Hecke eigenforms on  $\mathbb{H}_g$  of weight  $k$ . It is known that  $[\Gamma_4, 8]_0 = \mathbb{C} \cdot \varphi_4$  (for a nice proof of this, we refer to [5]). Poor [20] showed that  $\varphi_g(\tau)$  vanishes on the hyperelliptic locus  $\text{Hyp}_g$  for all  $g \geq 1$ , and the divisor of  $\varphi_g(\tau)$  in  $\mathcal{A}_g$  is proper and irreducible for all  $g \geq 4$ . And Ikeda [16, 17] proved that if  $g \equiv k \pmod{2}$ , there exists a canonical lifting

$$I_{g,k} : [\Gamma_1, 2k]_0 \longrightarrow [\Gamma_{2g}, g + k]_0.$$

Considering the special cases of the Ikeda lift maps  $I_{2,6}$  and  $I_{6,6}$ , Breulman and Kuss [2] showed that

$$I_{2,6}(\Delta) = c \varphi_4, \quad c(\neq 0) \in \mathbb{C},$$

and constructed a nonzero Siegel cusp form of degree 12 and weight 12 which is the image of  $\Delta(\tau)$ , where

$$\Delta(\tau) = (2\pi i)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q := e^{2\pi i \tau}, \quad \tau \in \mathbb{H}_1$$

is a cusp form of weight 12.

**Remark 5.3.** We consider a half-integral semi-positive symmetric integral matrix  $\mathcal{M}$  such that  $2\mathcal{M}$  is not even or which is not unimodular. The natural questions arise:

**Question 1.** Are there non-trivial stable Schottky-Jacobi forms of index  $\mathcal{M}$  for the universal Jacobian locus ?

**Question 2.** Are there non-trivial stable Schottky-Jacobi forms of index  $\mathcal{M}$  for the universal hyperelliptic locus ?

**Remark 5.4.** Let  $2\mathcal{M}$  be a positive even unimodular symmetric integral matrix of degree  $h$ . For two nonnegative integers  $k$  and  $\ell$ , we let  $A_{k,\ell\mathcal{M}}$  be the vector space of stable Jacobi forms of weight  $k$  and index  $\ell\mathcal{M}$ . We put

$$A(\mathcal{M}) := \bigoplus_{\ell=0}^{\infty} \bigoplus_{k=0}^{\infty} A_{k,\ell\mathcal{M}}.$$

Then we see easily that

$$A_{k,\ell\mathcal{M}} \bullet A_{p,q\mathcal{M}} \subset A_{k+p,(\ell+q)\mathcal{M}}$$

with respect to the natural multiplication  $\bullet$ . Thus  $A(\mathcal{M})$  is a bi-graded ring. Let  $I(\mathcal{M})$  be the space of all stable Schottky-Jacobi forms for the universal hyperelliptic locus contained in  $A(\mathcal{M})$ . Then  $I(\mathcal{M})$  is an ideal of  $A(\mathcal{M})$ .

According to Theorem 3.1, the subset

$$A(\mathcal{M})_0 := \bigoplus_{k=0}^{\infty} A_{k,0}$$

of  $A(\mathcal{M})$  has a polynomial ring structure.

Let

$$A^{[4]}(\mathcal{M})_1 := \bigoplus_{k \equiv 0 \pmod{4}} A_{k,\mathcal{M}}$$

and let  $B^{[4]}(\mathcal{M})_1$  be the subspace of all stable Schottky-Jacobi forms for the universal hyperelliptic locus contained in  $A^{[4]}(\mathcal{M})_1$ . Using Theorem 4.2 in [3], we can show that

$$(5.1) \quad (\Theta_P - \Theta_Q) \Theta_{2\mathcal{M}}$$

is a stable Schottky-Jacobi form for the universal hyperelliptic locus of weight  $\frac{1}{2}(m+h)$  and index  $\mathcal{M}$ , that is,  $(\Theta_P - \Theta_Q) \Theta_{2\mathcal{M}} \in B^{[4]}(\mathcal{M})_1$ . Here  $P$  and  $Q$  are two positive even unimodular quadratic forms of the same rank  $m$  ( $m \in \mathbb{Z}^+$ ), and  $\Theta_P, \Theta_Q$  are stable modular forms that are defined in Formula (3.1). The subspace  $B^{[4]}(\mathcal{M})_1$  of  $A^{[4]}(\mathcal{M})_1$  is spanned by all the stable Jacobi forms of type (5.1), where  $m$  runs over the set of all positive integers  $8n$  ( $n \in \mathbb{Z}^+$ ).

**Question 3.** What kinds of structures does  $A(\mathcal{M})$  have ?

## REFERENCES

- [1] W.L. Baily, *Satake's compactification of  $V_n^*$* , Amer. J. Math., **80** (1958), 348–364.
- [2] S. Breulmann and M. Kuss, *On a conjecture of Duke-Imamoğlu*, Proc. Amer. Math. Soc. **128** (2000), 1595–1604.
- [3] G. Codogni, *Hyperelliptic Schottky problem and stable modular forms*, Doc. Math. **21** (2016), 445–466.
- [4] G. Codogni and N. I. Shepherd-Barron, *The non-existence of stable Schottky forms*, Compos. Math. **150** (2014), no. 4, 679–690.
- [5] W. Duke and Ö. İmamoğlu, *Siegel modular forms of small weight*, Math. Ann. **310** (1998), 73–82.
- [6] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
- [7] E. Freitag, *Holomorphe Differentialformen zu Kongruenzgruppen der Siegelschen Modulgruppe*, Invent. Math. n. **30** (1975), 181–196.
- [8] E. Freitag, *Stabile Modulformen*, Math. Ann., **290** (1977), 197–211.
- [9] E. Freitag, *Die Irreduzibilität der Schottky-Relation (Bemerkung zu einem Satz von J. Igusa)*, Arch. der Math. **48** (1983), 255–259.
- [10] E. Freitag, *Siegelsche Modulfunktionen*, Springer (1983).
- [11] S. Grushevsky and R. Salvati Manni, *Jacobians with a vanishing theta-null in genus 4*, Israel J. Math., **164** (2008), 303–315.
- [12] S. Grushevsky and R. Salvati Manni, *The superstring cosmological constant and the Schottky form in genus 5*, Amer. J. Math., **133** (4) (2011), 1007–1027.
- [13] J. Harris and Hulek, *A remark on the Schottky locus in genus 4*, The Fano conference, 479–483, Univ. Torino, Turin, 2004.
- [14] J. Igusa, *On the irreducibility of Schottky divisor*, J. Fac. Sci. Tokyo **28** (1981), 531–545.
- [15] J. Igusa, *Schottky's invariant and quadratic forms*, E. B. Christoffel: the influence of his work on mathematics and the physical sciences, Birkhäuser, Basel (1981), 352–362.

- [16] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree  $2n$* , Ann. Math. **154** (2001), 641–681.
- [17] T. Ikeda, *Pullback of the lifting of elliptic cusp forms and Miyawaki’s conjecture*, Duke Math. J. **131** (2006), no. 3, 469–497.
- [18] H. Maass, *Über die Darstellung der Modulformen  $n$ -ten Grades durch Poincarésche Reihen*, Math. Ann. **123** (1951), 125–151.
- [19] H. Maass, *Siegel modular forms and Dirichlet series*, Lecture Notes in Math. **216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).
- [20] C. Poor, *Schottky’s form and the hyperelliptic locus*, Proc. Amer. Math. Soc. **124** (1996), 1987–1991.
- [21] I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc., **20** (1956), 259–281.
- [22] F. Schottky, *Zur Theorie der Abelschen Funktionen von vier Variablen*, J. Reine Angew. Math. **102** (1888), 304–352.
- [23] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Math., Vol. **7**, Springer-Verlag (1973).
- [24] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.
- [25] G. Shimura, *On certain reciprocity laws for theta functions and modular forms*, Acta Math. **141** (1979), 35–71.
- [26] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, edited by Jin-Woo Son and Jae-Hyun Yang, the Pyungsan Institute for Mathematical Sciences (1993), 33–58.
- [27] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.
- [28] J.-H. Yang, *Singular Jacobi Forms*, Trans. Amer. Math. Soc. **347** (6) (1995), 2041–2049.
- [29] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. **47** (6) (1995), 1329–1339.
- [30] J.-H. Yang, *Stable Automorphic Forms*, Proceedings of Japan-Korea Joint Seminar on Transcendental Number Theory and Related Topics, Masan, Korea (1998), 101–126.
- [31] J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree*, Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125–147 or Kyungpook Math. J. **40** (2) (2000), 209–237 or arXiv:math.NT/0602267.
- [32] J.-H. Yang, *Geometry and Arithmetic on the Siegel-Jacobi Space*, Geometry and Analysis on Manifolds, In Memory of Professor Shoshichi Kobayashi (edited by T. Ochiai, A. Weinstein et al), Progress in Mathematics, Volume **308**, Birkhäuser, Springer International Publishing AG Switzerland (2015), 275–325.
- [33] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg **59** (1989), 191–224.

DEPARTMENT OF MATHEMATICS, INHA UNIVERSITY, INCHEON 22212, KOREA  
*E-mail address:* `jhyang@inha.ac.kr`